9-4 INTERRECIPROCITY; SENSITIVITY TO SOURCES; HIGHER DERIVATIVES

Consider the network N and its adjoint $\hat{N}$ shown in Fig. 9-15. By Tellegen's theorem as given in Eqs. (2-36) and (2-37), the circuits satisfy

Not Differential →

$$\sum_{\substack{\text{all}\\\text{branches}}} (\hat{v}_k j_k - \hat{j}_k v_k) = 0 \tag{9-81}$$

so that

$$\sum_{\substack{\text{source}\\\text{branches}}} (\hat{v}_k j_k - \hat{j}_k v_k) = -\sum_{\substack{\text{internal}\\\text{branches}}} (\hat{v}_k j_k - \hat{j}_k v_k) \tag{9-82}$$

Let the hybrid internal branch relations of N be given by (9-56); then those of $\hat{N}$ will be given by (9-70). The right-hand side of (9-82) can be written as

$$\hat{\mathbf{j}}_B^T \mathbf{v}_B - \hat{\mathbf{v}}_B^T \mathbf{j}_B = \hat{\mathbf{j}}_{B1}^T \mathbf{v}_{B1} + \hat{\mathbf{j}}_{B2}^T \mathbf{v}_{B2} - \hat{\mathbf{v}}_{B1}^T \mathbf{j}_{B1} - \hat{\mathbf{v}}_{B2}^T \mathbf{j}_{B2} \tag{9-83}$$

When we use Eqs. (9-57), (9-63) and (9-58), (9-64), the right-hand side becomes

$$\hat{\mathbf{y}}^T \mathbf{x} - \hat{\mathbf{x}}^T \mathbf{y} = \hat{\mathbf{x}}^T \hat{\mathbf{H}}^T \mathbf{x} - \hat{\mathbf{x}}^T \mathbf{H} \mathbf{x} \tag{9-84}$$

The right-hand side of (9-84) is, by (9-68), identically zero. Hence for the special case of a network and its adjoint (9-82) can be more specific:

$$\sum_{\substack{\text{source}\\\text{branches}}} (\hat{v}_k j_k - \hat{j}_k v_k) = -\sum_{\substack{\text{internal}\\\text{branches}}} (\hat{v}_k j_k - \hat{j}_k v_k) = 0 \tag{9-85}$$

More restrictive than Tellegen's theorem

Two circuits which have this property are called *interreciprocal*.[4] Hence any network N and its adjoint network $\hat{N}$ form an interreciprocal pair.

Next, the interreciprocity relation (9-85) will be used to calculate the hitherto neglected effect of source variations on the output. Consider the special choice of sources made for $\hat{N}$ in Fig. 9-15b. From (9-85),

$$\sum_{\substack{l=1\\\text{voltage}\\\text{sources}}}^{k} (\hat{v}_l j_l - \hat{j}_l e_l) + \sum_{\substack{l=k+1\\\text{current}\\\text{sources}}}^{n} (\hat{v}_l i_l - \hat{j}_l v_l) + (\hat{v}_0 i_0 - \hat{i}_0 v_0) = -\sum_{l=1}^{k} \hat{j}_l e_l + \sum_{l=k+1}^{n} \hat{v}_l i_l - v_0$$

$$= 0 \tag{9-86}$$

Hence

$$v_0 = -\sum_{l=1}^{k} \hat{j}_l e_l + \sum_{l=k+1}^{n} \hat{v}_l i_l \tag{9-87}$$

and the desired sensitivities of v_0 to the source values are simply

Diff. gains:

$$\begin{aligned} \frac{\partial v_0}{\partial e_l} &= -\hat{j}_l \qquad l = 1, 2, \ldots, k \\ \frac{\partial v_0}{\partial i_l} &= \hat{v}_l \qquad l = k+1, k+2, \ldots, n \end{aligned} \tag{9-88}$$

Only $\hat{N}$ needs to be analyzed @ (Not N!) Superposition would require n analyses of N.

$$v_0 = \sum_l A_l e_l + \sum_k B_k i_k \qquad \frac{\partial v_0}{\partial e_i} = A_l$$

For $e_l = e_z = \ldots = 0$

$i_{k+1} = \ldots = 0$

$i_0 = \hat{i}_0 = 1A$

$$\hat{v}_0 i_0 - \hat{i}_0 v_0 = 0$$

$$z = \frac{v_0}{-i_0} = \hat{z} = \frac{\hat{v}_0}{-\hat{i}_0}$$

Thevenin equivalent of N can be found from 1 analysis of $\hat{N}$!

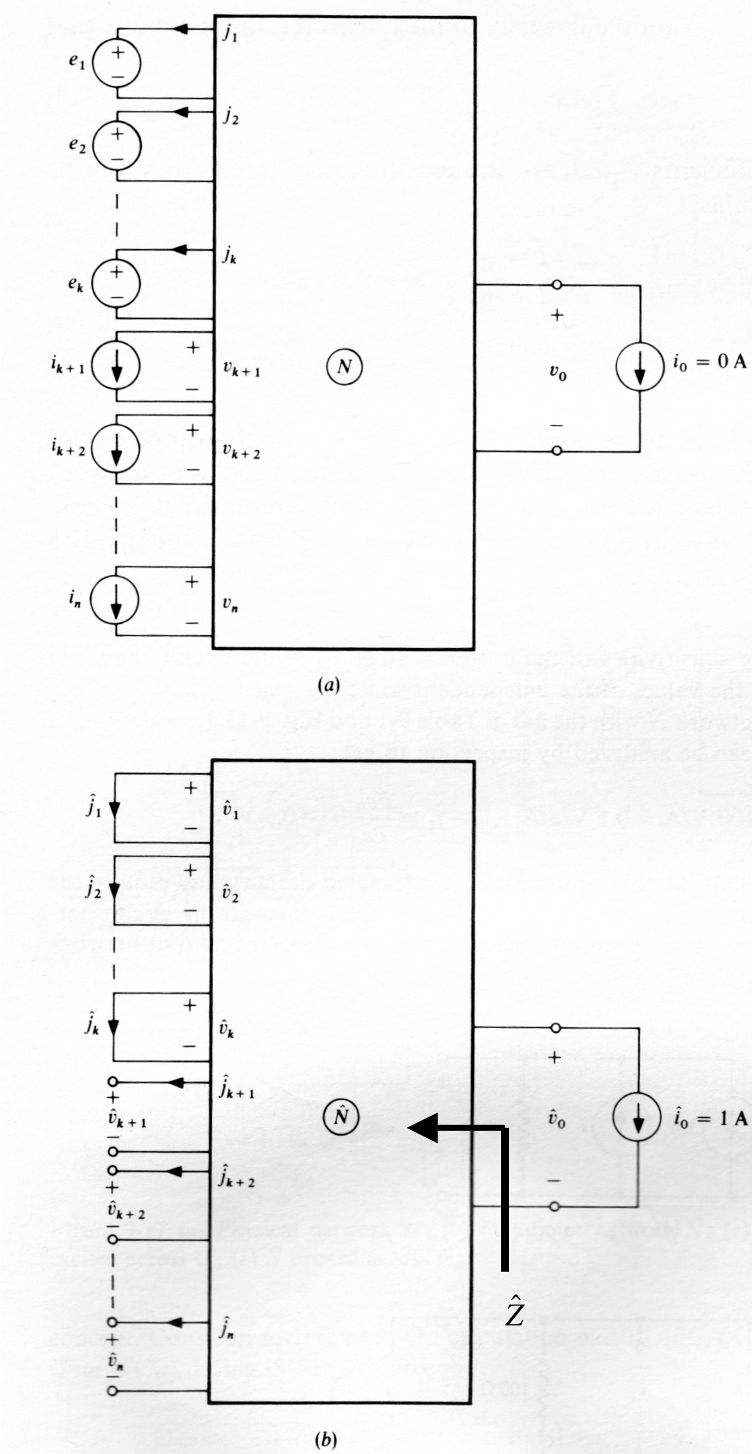
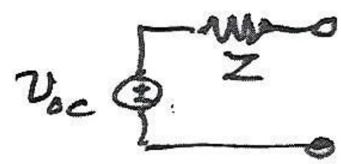


Figure 9-15 (*a*) Multisource circuit; (*b*) its adjoint network.



$$\hat{Z} = \frac{\hat{v}_o}{-1} = -\hat{v}_o = Z$$

R. Rohrer $N^2_{n_{out}} = \sum A_i^2 v_n^2$

Jssc uncorrelated noise sources v_{ni}

It should be noted that from the linearity of the circuit it follows directly that

$$v_0 = \sum_{l=1}^{k} A_l e_l + \sum_{l=k+1}^{n} B_l i_l \tag{9-89}$$

where the constant coefficients A_l, B_l are the sensitivities. Clearly, they can be obtained from the formulas:

Superposition:

$$A_m = v_0,\ e_l = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases} \qquad i_l = 0 \text{ for all } l$$
$$B_m = v_0,\ i_l = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases} \qquad e_l = 0 \text{ for all } l \tag{9-90}$$

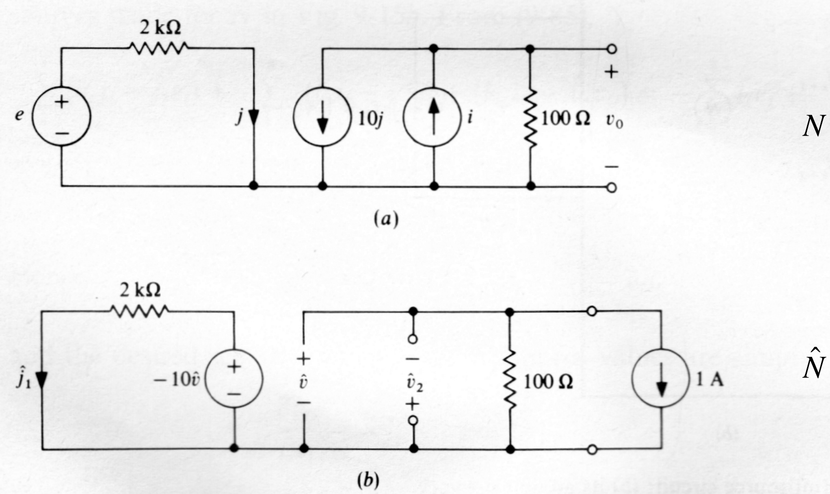
Thus, these sensitivities can be found at the cost of n single-source circuit analyses without recourse to the adjoint network $\hat{N}$. Since the adjoint network approach requires only *one* analysis of the single-source circuit $\hat{N}$, it is more economical even for a two-source circuit; it becomes imperative for circuits with many ($n > 10$) independent sources.

Example 9-9 Calculate the sensitivities of the output voltage v_0 in the circuit shown in Fig. 9-16*a* to variations in the values of the independent sources e and i.

Drawing the adjoint network $\hat{N}$ with the aid of Table 9-1 and Fig. 9-15 gives the circuit of Fig. 9-16*b*. This circuit can be analyzed by inspection to get

$$\hat{v}_2 = \frac{\partial v_0}{\partial i} = 100 \text{ V/A} = 0.1 \text{ V/mA} \qquad -\hat{j}_1 = \frac{\partial v_0}{\partial e} = -0.5 \text{ V/V}$$

It will be shown in Chap. 11 that optimization (automated design) may require the calculation of the second partial derivatives of the output with respect to the circuit parameters. These derivatives can also be found efficiently with the aid of the adjoint-network



$v_0 = -\hat{j}_1 e + \hat{v}_2 i$

Figure 9-16 (*a*) Circuit with two independent sources; (*b*) its adjoint network.

$\frac{\partial^2 v_0}{\partial x_1 \partial x_2}$ can also be found. Used in circuit optimization, in Hessian matrix.